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One-Dimensional Diffusion in a Semiinfinite Poisson Random Force

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We consider the one-dimensional diffusion of a particle on a semiinfinite line and in a piecewise linear random potential. We first present a new formalism which yields an analytical expression for the Green function of the Fokker–Planck equation, valid for any deterministic construction of the potential profile. The force is then taken to be an asymmetric dichotomic process. Solving the corresponding energy-dependent stochastic Riccati equation in the space-asymptotic regime, we give an exact probabilistic description of returns to the origin. This method allows for a time-asymptotic characterization of the underlying dynamical phases. When the two values taken by the dichotomic force are of different signs, there occur trapping potential wells with a broad distribution of trapping times, and dynamical phases may appear, depending on the man force. If both values are negative, the time-asymptotic mean value of the probability density at the origin is proportional to the absolute value of the mean force. If they are both positive, traps no longer exist and the dynamics is always normal. Problems with a shot-noise force and with a Gaussian white-noise force are solved as appropriate limiting cases.

KEY WORDS: Fluctuation phenomena; random processes; Brownian motion.

1. INTRODUCTION

The study of dynamical features of diffusion or conductivity in random environments has been initiated in early eighties[(1)] and since then intensively

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pursued.[3–7] The problem is usually formulated either in a discrete form (c.f., a comprehensive exposition in ref. 7, Section 6), i.e., by means of the Pauli Master Equation,[8] assuming the transfer rates to be random variables, or in a continuous formulation (c.f., a survey of results in ref. 6 and the references therein), i.e., using the Fokker–Planck equation[9–11] and assuming the so-called drift-function to be a stochastic function in space. The transport properties are obtained through an average over the random parameters in the equation of motion.

The resulting dynamics in the time-asymptotic region may exhibit nonstandard features. For instance, with a Gaussian drift function of non-zero mean value, a succession of dynamical phases is observed when the bias is varied. These phases are characterized by different drift and diffusion behaviours, normal (i.e., with a finite mobility and diffusion coefficient), or not. The existence of anomalous dynamical phases can be traced back to the existence of traps with a broad distribution of trapping times.[5]

Most of the standard treatments assume independent random transfer rates in the master equation,[7, 12, 13] or, correspondingly, a white-noise drift function in the Fokker–Plank equation.[14–17] In the latter case, the problem can be referred to as the diffusion process in a Brownian environment. Assuming the continuous formulation and the positionally independent random properties of the medium, the problem has been treated both in physical (c.f., for example, ref. 16) and in mathematical (e.g., ref. 6) literature. Several modifications of the continuous model have been analyzed (differing by the boundary conditions, with or without the global bias) using various methods and levels of rigorousness. Nevertheless, as pointed out, e.g., in ref. 18, a more realistic description should take into account the possibility of spatial correlations in the local transport parameters.[19–23]

Let us consider a continuous medium and imagine first that the particle diffuses through an array of randomly positioned minima with randomly distributed depths. The simplest choice for a space-correlated bias yielding this scenario is that of a dichotomic noise,[11] with realizations alternately assuming two possible values of different signs. The potential then displays a succession of linear segments of random lengths and of alternately positive and negative slopes. One can well figure out that, in such a potential, the minima represent traps, some of them being deep traps with large trapping times. This case shares some similarities with the Gaussian-white-noise one, and should lead to results of the same type for the particle drift and diffusion properties (i.e., the existence of anomalous dynamical phases for certain values of the parameters of the model). But, interestingly enough, the choice of a dichotomic noise also allows to treat the case of quenched random force taking alternately two values of the same sign. Since clearly the notion of deep traps—and even simply of traps—makes no more sense in this case, one does not expect the existence of anomalous dynamical phases.

More specifically, in the present paper, we assume that the lengths of the above described constant-force segments are independent and exponentially distributed, in which case the quenched random force is a Markovian Poisson process in space. We consider a particle diffusion on a semi-infinite line, i.e., we impose a reflecting boundary condition at the origin. These two hypotheses allow for exact analytical calculation and for the discussion of a rich variety of physically different situations.

The paper is organized as follows. In Section 2, our analysis begins with the Laplace transformation of the Fokker–Planck equation. One thus gets a differential equation in space, depending parametrically on the energy, as pictured by the Laplace variable z. the corresponding Green function in the presence of an arbitrary deterministic piecewise constant bias is derived. In Section 3, we introduce quenched disorder with general piecewise constant realizations. The localization probability of the particle at its (sharp) initial position satisfies a Riccati stochastic differential equation with a multiplicative noise (in space). Specifying further the noise to be the Markovian Poisson one, we are then able to derive a one-formula based (Eq. (50)) parallel analysis of the disorder average of the localization probability of the particle at the origin on the one hand and the trapping time or the time-asymptotic average velocity on the other hand. In Section 4, we analyze various physical situations, according to the sign of the mean bias and to the presence or to the absence of traps. Finally, Section 5 contains our conclusions.

Generally speaking, the new results of our paper are the following. First, our procedure is exact for any fixed value of the Laplace variable z. On the one hand, this enables, at least in principle, the analysis of the disorder-averaged probability density at the origin for *any* time. On the other hand, we can carry out the small-z analysis and derive the exact time-asymptotic formulae for this quantity in various physical regimes. Another specific feature of our work is the application of reflecting boundary conditions at the origin. Thus, clearly, in contrast to the equivalent problem on an infinite line, the situation with a positive or a negative mean bias *are not* equivalent. Indeed, with a negative mean bias the particle is in some sense stuck to the boundary at the origin or pushed back towards it, while with a positive mean bias it escapes towards infinity, the *modus* of its time-asymptotic motion being controlled by a parameter describing the typical depth of the potential traps.

2. DIFFUSION IN A DETERMINISTIC FORCE

2.1. Homogeneous Force

Let us consider an overdamped Brownian particle acted upon by a standard white-noise Langevin force $\Gamma(\tilde{t})$ and by a position-dependent potential force $F(\tilde{x})$. Its dynamics is described by the viscous Langevin equation

$$\eta \frac{d}{d\tilde{t}} \tilde{x}(\tilde{t}) = F[\tilde{x}(\tilde{t})] + \Gamma(\tilde{t}) \tag{1}$$

with η being the viscosity. The correlation function of the Langevin force is equal to $2D_0\eta^2\delta(\tilde{t}-\tilde{t}')$, where $D_0 = k_B T/\eta$ is the diffusion constant in the absence of the potential force. The corresponding Fokker–Planck equation for the Green function $\tilde{P}(\tilde{x}, \tilde{y}; \tilde{t})$ reads

$$\frac{\partial}{\partial \tilde{t}} \tilde{P}(\tilde{x}, \tilde{y}; \tilde{t}) = -\frac{\partial}{\partial \tilde{x}} \left[-D_0 \frac{\partial}{\partial \tilde{x}} \tilde{P}(\tilde{x}, \tilde{y}; \tilde{t}) + \frac{F(\tilde{x})}{\eta} \tilde{P}(\tilde{x}, \tilde{y}; \tilde{t}) \right] \tag{2}$$

The bracketed expression represents the probability current $\tilde{J}(\tilde{x}, \tilde{y}; \tilde{t})$. We assume the initial condition $\tilde{P}(\tilde{x}, \tilde{y}; 0) = \delta(\tilde{x}-\tilde{y})$ and the boundary conditions $\tilde{J}(\tilde{x}_0, \tilde{y}; t) = 0$, $\tilde{J}(\tilde{x}_1, \tilde{y}; \tilde{t}) = 0$. Consequently, the boundaries at $\tilde{x}_0$ and $\tilde{x}_1$ are reflecting and the probability density is always normalized to unity.

In order to make the following calculation more transparent, we introduce dimensionless variables. The potential force will be written in the form $F(\tilde{x}) = F_0 f(\tilde{x})$. The dimensionless coordinate is $x = \tilde{x}F_0/\eta D_0$, and the dimensionless time $t = \tilde{t}F_0^2/\eta^2 D_0$. We thus get from Eq. (2):

$$\frac{\partial}{\partial t} P(x, y; t) = -\frac{\partial}{\partial x} J(x, y; t) \tag{3}$$

$$J(x, y; t) = -\frac{\partial}{\partial x} P(x, y; t) + f(x)\, P(x, y; t) \tag{4}$$

The original density and current are connected with their dimensionless counterparts via $\tilde{P}(\tilde{x}, \tilde{y}; \tilde{t}) = F_0 P(x, y; t)/\eta D$ and $\tilde{J}(\tilde{x}, \tilde{y}; \tilde{t}) = F_0^2 J(x, y; t)/\eta^2 D$.

Let us assume for the moment a position-independent force: $f(x) = f$ for $x \in [x_0, x_1]$. Performing the Laplace transformation, the Fokker–Planck equation (2) yields a nonhomogeneous differential equation with constant coefficients,

$$\left[\frac{d^2}{dx^2} - f \frac{d}{dx} + z \right] P(x, y; z) = -\delta(x-y) \tag{5}$$

We are using the same symbol for a given function $a(t)$ and for its Laplace transform $a(z)=\int_0^\infty dt \exp(-zt)\, a(t)$, the Laplace original (transform) being always indicated by writing the variable $t(z)$. Combining any particular solution of the nonhomogeneous equation with the general solution of the homogeneous equation, we have $P(x, y; z)=P_N(x, y: z)+P_H(x; z)$, with

$$P_N(x, y; z)=\frac{1}{2\alpha(z)}\,\{\Theta(y-x)\,\mathrm{e}^{(x-y)\,\alpha^+(z)}+\Theta(x-y)\,\mathrm{e}^{-(x-y)\,\alpha^-(z)}\} \tag{6}$$

$$P_H(x; z)=c^+(z)\,\mathrm{e}^{x\alpha^+(z)}+c^-(z)\,\mathrm{e}^{-x\alpha^-(z)} \tag{7}$$

where $\alpha(z)=\sqrt{z+f^2/4}$, $\alpha^\pm(z)=\sqrt{z+f^2/4}\pm f/2$, and $\Theta(x)$ is the Heaviside function. Having acquired the general solution for the density, we calculate the general expression for the probability current. In the last step, the two functions $c^\pm(z)$ are fixed from the reflecting-boundary conditions $J(x_0, y; z)=0$, $J(x_1, y; z)=0$.

The whole procedure is well known. However, in view of the following calculation, it is convenient to present the final result in a matrix form. We introduce a two-dimensional space with the basis $\{|1, 0\rangle, |0, 1\rangle\}$ and we express the pair density-current as two coordinates of a single state ket: $P(x, y; z)=\langle 1, 0 \mid G(x, y; z)\rangle$ and $J(x, y; z)=\langle 0, 1 \mid G(x, y; z)\rangle$. Adopting this convention, the result of the present simple example reads

$$|G(x, y; z)\rangle=\mathsf{W}(x_1-x; z)\,|1, 0\rangle\,\Gamma(y; z)-\Theta(y-x)\,\mathsf{W}(y-x; z)\,|0, 1\rangle \tag{8}$$

Here we have introduced the abbreviation

$$\Gamma(y; z)=\frac{\langle 0, 1|\,\mathsf{W}(y-x_0; z)\,|0, 1\rangle}{\langle 0, 1|\,\mathsf{W}(x_1-x_0; z)\,|1, 0\rangle} \tag{9}$$

and the matrix

$$\mathsf{W}(x; z)=\begin{pmatrix} \dfrac{\alpha^-(z)\,\mathrm{e}^{x\alpha^-(z)}+\alpha^+(z)\,\mathrm{e}^{-x\alpha^+(z)}}{2\alpha(z)} & \dfrac{\mathrm{e}^{x\alpha^-(z)}-\mathrm{e}^{-x\alpha^+(z)}}{2\alpha(z)} \\ z\,\dfrac{\mathrm{e}^{x\alpha^-(z)}-\mathrm{e}^{-x\alpha^+(z)}}{2\alpha(z)} & \dfrac{\alpha^+(z)\,\mathrm{e}^{x\alpha^-(z)}+\alpha^-(z)\,\mathrm{e}^{-x\alpha^+(z)}}{2\alpha(z)} \end{pmatrix} \tag{10}$$

Notice that, at given z, $\mathsf{W}(x; z)$ satisfies the "dynamical equation"

$$\frac{d}{dx}\mathsf{W}(x; z) = \mathsf{H}(z)\,\mathsf{W}(x; z), \qquad \mathsf{H}(z) = \begin{pmatrix} -f & 1 \\ z & 0 \end{pmatrix} \tag{11}$$

with the position x playing here the role of time. Eq. (10) gives $\mathsf{W}(0; z) = \mathsf{I}$ (unity matrix), i.e., we have formally $\mathsf{W}(x; z) = \exp[x\mathsf{H}(z)]$.

Equation (8) yields the complete picture of the resulting motion (i.e., we can compute $P(x, y; z)$, $J(x, y; z)$ and these functions can be inverted into the time domain[26]). For instance, taking $x_0 = 0$, $x_1 = l$, $y \to 0^+$, and $x = 0$, the probability density at the origin emerges as the ratio of two matrix elements:

$$P(0, 0; z) = \frac{\langle 1, 0|\,\mathsf{W}(l; z)\,|1, 0\rangle}{\langle 0, 1|\,\mathsf{W}(l; z)\,|1, 0\rangle} = \frac{1}{z}\,\frac{\alpha^-(z)\,e^{l\alpha^-(z)} + \alpha^+(z)\,e^{-l\alpha^+(z)}}{e^{l\alpha^-(z)} - e^{-l\alpha^+(z)}} \tag{12}$$

Moreover, for the semi-infinite line, we have $\lim_{l\to\infty} P(0, 0; z) = \alpha^-(z)/z$. In this case, we get following picture. having $f < 0$, the force pushes the diffusing particle against the boundary. In this case, the time-asymptotic value of the probability density at the origin is $|f|$ the asymptotic value of the mean particle position is $|f|^{-1}$, i.e., the time-asymptotic velocity is zero. On the other hand, when $f > 0$, $\lim_{l\to\infty} P(0, 0; t)$ decreases exponentially to zero and the mean position increases linearly with time, the velocity being just f. Finally, in the marginal case $f = 0$, $P(0, 0; t)$ behaves asymptotically as $1/\sqrt{\pi t}$, the mean position increases as $\sqrt{\pi t}$, and the asymptotic velocity is zero.

2.2. Piecewise Constant Force

Let the original interval $[x_0, x_N]$ be divided into N segments $[x_{k-1}, x_k]$, $k = 1,\ldots, N$, with lengths $l_k = x_k - x_{k-1}$. Let f_k be the constant force in the kth subinterval. We assume that the particle has been originally placed in the Mth segment, i.e., $P(x, y; 0) = \delta(x - y)$ with $y \in [x_{M-1}, x_M]$. The boundary conditions at x_0 and x_N are again reflecting.

The procedure for solving the Fokker–Planck equation will be parallel to that in the above simple example. The general solution in the Mth segment consists of two parts. First, the particular solution of the non-homogeneous Eq. (5) (with f_M instead of f) will assume the form (6) with $\alpha_M(z) = \sqrt{z + f_M^2/4}$ instead of $\alpha(z)$ and $\alpha_M^\pm(z) = \alpha_M(z) \pm f_M/2$ instead of $\alpha^\pm(z)$. Second, the general solution of the homogeneous equation in the Mth subinterval assumes the form (7), again with $\alpha_M^\pm(z)$ instead of $\alpha^\pm(z)$, and with two arbitrary functions $c_M^{\pm}(z)$ instead of $c^{\pm}(z)$. The general solution in the kth subinterval, $k \neq M$, is also of the form (7) with the substitutions $\alpha^{\pm}(z) \to \alpha_k^{\pm}(z)$ and $c^{\pm}(z) \to c_k^{\pm}(z)$. Altogether, the whole general solution depends on the $2N$ functions $c_k^{\pm}(z)$, $k = 1,..., N$. these are fixed from the requirements $J(x_0, y; z) = J(x_N, y; z) = 0$ at the reflecting boundaries and from the continuity conditions for the probability density and for the probability current at the intermediate points $x_1, x_2,..., x_{N-1}$.

The final result can be again expressed in matrix form. We designate the "evolution operator" for the kth segment as $\mathsf{W}_k(x; z)$—it is defined by the expression (10) with the substitutions $\alpha(z) \to \alpha_k(z)$ and $\alpha^{\pm}(z) \to \alpha_k^{\pm}(z)$. Further, we introduce the notations

$$\mathsf{W}_{m,n} = \mathsf{W}_m(l_m; z)\, \mathsf{W}_{m+1}(l_{m_1}; z) \cdots \mathsf{W}_n(l_n; z), \tag{13}$$

$$\Gamma_{M,N}(y; z) = \frac{\langle 0, 1|\, \mathsf{W}_{1,M-1} \mathsf{W}_M(y - x_{M-1}; z)\, |0, 1\rangle}{\langle 0, 1|\, \mathsf{W}_{1,N}\, |1, 0\rangle} \tag{14}$$

Finally, let $|G_k(x, y; z)\rangle$ be the value of the state ket in the kth segment, that is $|G(x, y; z)\rangle = |G_k(x, y; z)\rangle$ for $x \in [x_{k-1}, x_k]$. The final result of the present Section then reads

$$\begin{aligned} |G_1(x, y; z)\rangle = {} & \mathsf{W}_1(x_1 - x; z)\, \mathsf{W}_{2,N}\, |1, 0\rangle\, \Gamma_{M,N}(y; z) \\ & - \mathsf{W}_1(x_1 - x; z)\, \mathsf{W}_{2,M-1} \mathsf{W}_M(y - x_{M-1}; z)\, |0, 1\rangle \end{aligned} \tag{15}$$

$$\vdots$$

$$\begin{aligned} |G_{M-1}(x, y; z)\rangle = {} & \mathsf{W}_{M-1}(x_{M-1} - x; z) \mathsf{W}_{M,N}\, |1, 0\rangle\, \Gamma_{M,N}(y; z) \\ & - \mathsf{W}_{M-1}(x_{M-1} - x; z)\, \mathsf{W}_M(y - x_{M-1}; z)\, |0, 1\rangle \end{aligned} \tag{16}$$

$$\begin{aligned} |G_M(x, y; z)\rangle = {} & \mathsf{W}_M(x_M - x; z)\, \mathsf{W}_{M+1,N}\, |0, 1\rangle\, \Gamma_{M,N}(y; z) \\ & - \Theta(y - x)\, \mathsf{W}_M(y - x; z)\, |0, 1\rangle \end{aligned} \tag{17}$$

$$|G_{M+1}(x, y; z)\rangle = \mathsf{W}_{M+1}(x_{M+1} - x; z)\, \mathsf{W}_{M+2,N}\, |1, 0\rangle\, \Gamma_{M,N}(y; z) \tag{18}$$

$$\vdots$$

$$|G_N(x, y; z)\rangle = \mathsf{W}_N(x_N - x; z)\, |1, 0\rangle\, \Gamma_{M,N}(y; z) \tag{19}$$

It is easy to check that the boundary conditions and the continuity conditions are actually satisfied and that the resulting probability density is properly normalized.

Having at hand the complete Green function for the given composition of the segments, we now proceed to the analysis of some consequences.

First, let $x_0 = 0$, $x_N = l$, $y \to 0^+$, and $x = 0$. The probability density at the origin assumes a particularly simple form:

$$P(0, 0; z) = \frac{\langle 1, 0|\, \mathsf{W}_1(l_1; z)\, \mathsf{W}_2(l_2; z) \cdots \mathsf{W}_N(l_N; z)\, |1, 0\rangle}{\langle 0, 1|\, \mathsf{W}_1(l_1; z)\, \mathsf{W}_2(l_2; z) \cdots \mathsf{W}_N(l_N; z)\, |1, 0\rangle} \tag{20}$$

In Section 3, this expression will be taken as a starting point for the disordered-medium calculation.

Second, consider the value of the probability density at point y in the Mth segment, i.e., at the initial position of the particle. Taking $x_0 = 0$ and $x \to y^+$, $P(x, x; z)$ equals the ratio

$$\frac{\langle 0, 1|\, \mathsf{W}_{1, M-1} \mathsf{W}_M(x - x_{M-1}; z)\, |0, 1\rangle \langle 1, 0|\, \mathsf{W}_M(x_M - x; z)\, \mathsf{W}_{M+1, N}\, |1, 0\rangle}{\langle 0, 1|\, \mathsf{W}_1(l_1; z) \cdots \mathsf{W}_M(l_M; z) \cdots \mathsf{W}_N(l_N; z)\, |1, 0\rangle} \tag{21}$$

Due to the exponential nature of the operator $\mathsf{W}_M(l_M; z)$ in the denominator, we can split it as a product of two factors, $\mathsf{W}_M(x - x_{M-1}; z)\, \mathsf{W}_M(x_M - x; z)$. Further, we can insert in between the resolution of the unity operator, which yields

$$P(x, x; z) = \left[\frac{\langle 0, 1|\, \mathsf{W}_k(x_k - x; z)\, \mathsf{W}_{k+1}(l_{k+1}; z) \cdots \mathsf{W}_N(l_N; z)\, |1, 0\rangle}{\langle 1, 0|\, \mathsf{W}_k(x_k - x; z)\, \mathsf{W}_{k+1}(l_{k+1}; z) \cdots \mathsf{W}_N(l_N; z)\, |1, 0\rangle} \right.$$
$$\left. + \frac{\langle 0, 1|\, \mathsf{W}_1(l_1; z) \cdots \mathsf{W}_{k-1}(l_{k-1}; z)\, \mathsf{W}_k(x - x_{k-1}; z)\, |1, 0\rangle}{\langle 0, 1|\, \mathsf{W}_1(l_1; z) \cdots \mathsf{W}_{k-1}(l_{k-1}; z)\, \mathsf{W}_k(x - x_{k-1}; z)\, |0, 1\rangle} \right]^{-1} \tag{22}$$

For any *finite* x, the small-z limit of the second term turns out to be zero. Thus the quantity $\lim_{z \to 0^+} P(x, x; z)$ is equal to the small-z limit of the density at origin for a *new* medium of the total length $l - x$. The new interval consists of $N - k + 1$ segments of the lengths $x_k - x$, $l_{k+1}, \ldots, l_N$, the constant forces in these segments being $f_k, \ldots, f_N$. This simple result will be of considerable value in the analysis of the disordered medium. Namely, imagine that the particle is launched from the point x. The total time $T(x)$ spent in the interval $(x, x + dx)$ is equal to

$$T(x)\, dx = \left[\int_0^\infty P(x, x; t)\, dt \right] dx = \lim_{z \to 0^+} P(x, x; z)\, dx \tag{23}$$

i.e., $T(x)$ can be related to the probability density at the origin for the above mentioned new interval. Of course, $T(x)$ can only be finite for an interval of infinite length. However, even for the semi-infinite interval, $T(x)$ is finite only if the particle can escape to infinity, that is, if the force in the last segment (which is necessarily of infinite length) is non-negative.

Third, let us consider the thermally-averaged position of the particle $M(l; t)$ (we take again $x_0 = 0$, $x_N = l$, and $y \to 0^+$). Introducing the Laplace transform $M(l; z) = \int_0^l dx\, xP(x, 0; z)$ and integrating the Laplace transform of the Fokker–Planck equation, we have

$$\begin{aligned} M(l; z) &= \left[x \int_0^x dx'\, P(x', 0; z) \right]_0^l - \int_0^l dx \int_0^x dx'\, P(x', 0; z) \\ &= \frac{l}{z} - \int_0^l dx\, \frac{1}{z} \left[1 - J(x, 0; z) \right] = \frac{1}{z} \int_0^l J(x, 0; z)\, dx \end{aligned} \tag{24}$$

Thereupon, taking the projection of the above Green function, we can write

$$M(l; z) = \frac{1}{z} \frac{\left[\begin{array}{c} \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \langle 0, 1 |\, \mathsf{W}_k(x_k - x; z)\, \mathsf{W}_{k+1}(l_{k+1}; z) \\ \cdots \mathsf{W}_N(l_N; z)\, | 0, 1 \rangle\, dx \end{array} \right]}{\langle 0, 1 |\, \mathsf{W}_1(l_1; z)\, \mathsf{W}_2(l_2; z) \cdots \mathsf{W}_N(l_N; z)\, | 0, 1 \rangle} \tag{25}$$

In the next section, this result will allow to connect the thermally-averaged position $M(l; z)$ with the Laplace transform $P(0, 0; z)$ of the probability density at the origin. We now turn to the detailed analysis of this latter quantity in the disordered medium.

3. PIECEWISE CONSTANT RANDOM FORCE

The preceding calculation in valid for any *deterministic* piecewise constant force. Such a force can be conceived as a member of a randomly constructed family of functions, that is, as a realization of a stochastic process. An arbitrary fixed realization is associated with a given weight. Consequently, the same weight is attributed to the probability density at the origin $P(0, 0; z)$ for this realization. We are thus guided to the question: what is the probability density of the random variable $P(0, 0; z)$? This section presents the exact answer for a semi-infinite medium and a special type of the stochastic process to be fully specified in Subsection 3.3. In the first and the second Subsection, our reasoning is valid for *any* piecewise constant random force.

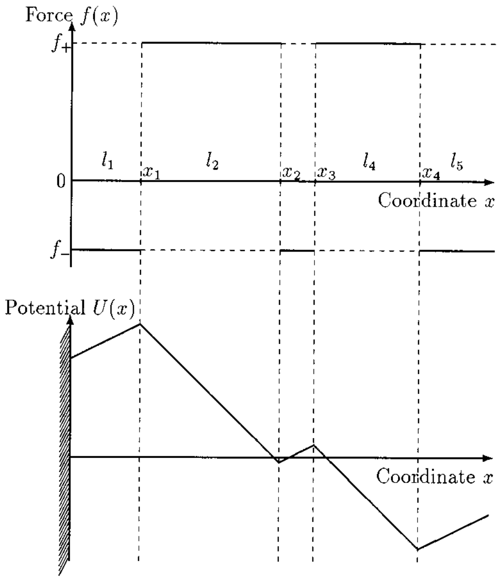

Fig. 1. Typical realization of the Markovian Poisson dichotomic force taking two values of opposite signs, $f_- < 0 < f_+$ with $f_+ = 2\,|f_-|$. The slope of the corresponding potential is alternately positive and negative: there are traps in this case.

3.1. Stochastic Riccati Equation

Let us consider the following system of two stochastic differential equations,

$$\frac{d}{d\lambda}\,|\psi(\lambda; z)\rangle = \mathsf{H}(\lambda; z)\,|\psi(\lambda; z)\rangle, \qquad \mathsf{H}(\lambda; z) = \begin{pmatrix} -\phi(\lambda) & 1 \\ z & 0 \end{pmatrix} \tag{26}$$

where $\phi(\lambda)$, $\lambda \geqslant 0$, is a piecewise constant random process. Let us introduce the projections $R(\lambda; z) = \langle 1, 0 \mid \psi(\lambda; z)\rangle$, $S(\lambda; z) = \langle 0, 1 \mid \psi(\lambda; z)\rangle$, and take the initial conditions $R(0; z) = 1$, $S(0; z) = 0$. The system (26) can be solved for any specific realization of $\phi(\lambda)$. Actually, consider the composition described at the beginning of Subsection 2.2 (c.f. also Fig. 1), and let the "time" λ equal $l - x$. We have $\phi(\lambda; z) = f_N$ for $\lambda \in [0, l_N]$. If λ increases, the evolution is controlled by the operator $\mathsf{W}_N(\lambda; z)$. At the end of this interval, the state $\mathsf{W}_N(l_N; z)\,|1, 0\rangle$ represents the initial condition for the semigroup evolution in the succeeding interval $\lambda \in [l_N, l_{N-1} + l_N]$. This evolution is governed by the operator $\mathsf{W}_{N-1}(\lambda; z)$. Repeating this reasoning, the solution at the end of the Nth interval, i.e., at the point $\lambda = 1$, reads

$$R(l; z) = \langle 1, 0|\, \mathsf{W}_1(l_1; z)\, \mathsf{W}_2(l_2; z) \cdots \mathsf{W}_N(l_N; z)\, |1, 0\rangle \tag{27}$$

$$S(l; z) = \langle 0, 1|\, \mathsf{W}_1(l_1; z)\, \mathsf{W}_2(l_2; z) \cdots \mathsf{W}_N(l_N; z)\, |1, 0\rangle \tag{28}$$

By comparing with Eq. (20), it can be seen that the random variable $P(0, 0; z)$ is identical to the ratio $R(l; z)/S(l; z)$. From now on, the random variable $P(0, 0; z)$ will be designated as $P(\lambda; z)$. We introduce its probability density $\pi(p, \lambda; z) = \langle \delta[p - P(\lambda; z)]\rangle$, where the symbol $\langle \cdots \rangle$ denotes the average over the quenched disorder. Taking the projections of Eq. (26), the random variables $R(\lambda; z)$ and $S(\lambda; z)$ obey the system of stochastic differential equations

$$\frac{d}{d\lambda} R(\lambda; z) = -\phi(\lambda)\, R(\lambda; z) + S(\lambda; z), \qquad \frac{d}{d\lambda} S(\lambda; z) = zR(\lambda; z) \tag{29}$$

with λ representing the total length of the interval accessible to the diffusing particle. Finally, on comparing the λ-derivative of the product $R(\lambda; z) = P(\lambda; z)\, S(\lambda; z)$ with the first Eq. (29) and on dividing by $S(\lambda; z)$, we obtain the stochastic Riccati differential equation obeyed by $P(\lambda; z)$:

$$\frac{d}{d\lambda} P(\lambda; z) = -zP^2(\lambda; z) - \phi(\lambda)\, P(\lambda; z) + 1, \qquad P(0; z) = +\infty \tag{30}$$

For purely operational reasons, we introduce the random variable $Q(\lambda; z) = zP(\lambda; z)$. One gets from Eq. (30):

$$\frac{d}{d\lambda} Q(\lambda; z) = -Q^2(\lambda; z) - \phi(\lambda)\, Q(\lambda; z) + z, \qquad Q(0; z) = +\infty \tag{31}$$

The density $\kappa(q, \lambda; z) = \langle \delta[q - Q(\lambda; z)]\rangle$ is related with the density $\pi(p, \lambda; z)$ by $\pi(p, \lambda; z) = z\kappa(zp, \lambda; z)$. Regarding the relation

$$\lim_{t \to \infty} P(0, 0; t) = \lim_{z \to 0^+} zP(\lambda; z) = \lim_{z \to 0^+} Q(\lambda; z) \tag{32}$$

the small-z limit of the density $\kappa(q, \lambda; z)$ describes the time-asymptotic properties of the random variable $P(0, 0; t)$.

3.2. Mean Trapping Time and Mean Velocity

Let us now return to the reasoning associated with Eqs. (21)–(23). We shall call the variable $T(x) = \lim_{z \to 0^+} P(x, x; z)$ the *trapping time*.[3]

[3] We shall use this term even if the potential is monotonous, although a more appropriate name in this case would be the *dwelling time*.

Presently, the density $P(x, x; z)$ and hence also the trapping time are random variables. Performing the small-z limit in Eq. (22), we have connected the trapping time with the particle-position probability density at the origin of an interval shorter than the original one. However, if the original interval is of *infinite* length, the same is true for the new one. Since the quenched random force is described by a *stationary* process, the probability density at the beginning of the new interval is stochastically equivalent with the density at the beginning of the original one. Summing up, one has:

$$T(x) = \lim_{z \to 0^+} \lim_{\lambda \to \infty} P(x, x; z) = \lim_{z \to 0^+} \lim_{\lambda \to \infty} P(\lambda; z) \tag{33}$$

The trapping time is position-independent, i.e., $T(x) = T$, its mean value being

$$\tau \stackrel{\text{def}}{=} \lim_{z \to 0^+} \lim_{\lambda \to \infty} \langle P(\lambda; z) \rangle = \lim_{z \to 0^+} \lim_{\lambda \to \infty} \frac{1}{z} \langle Q(\lambda; z) \rangle \tag{34}$$

In the preceding section, we have introduced the Laplace transform of the thermally-averaged position, $M(\lambda; z)$. Presently, it is again a random variable. Using the definition of $S(\lambda, z)$, the probability current in Eq. (25) can be rewritten as $J(x, 0; z) = S(\lambda - x; z)/S(\lambda; z)$, which yields

$$M(\lambda; z) = \frac{1}{z} \int_0^\lambda dx\, J(x, 0; z) = \frac{1}{z} \frac{\int_0^\lambda dx\, S(x; z)}{S(\lambda; z)} \tag{35}$$

where we have used the stationarity of the quenched random force. The derivative of this equation yields

$$\frac{d}{d\lambda} M(\lambda; z) = \frac{1}{z} - zP(\lambda; z)\, M(\lambda; z), \quad M(0; z) = 0 \tag{36}$$

Finally, introducing the Laplace transform of the thermally-averaged velocity, $V(\lambda; z) = zM(\lambda; z)$, the *time*-asymptotic velocity for the semi-infinite line is given by

$$\lim_{t \to \infty} \lim_{\lambda \to \infty} V(\lambda; t) = \lim_{z \to 0^+} \lim_{\lambda \to \infty} z \int_0^\lambda d\lambda' \exp\left[-z \int_{\lambda'}^\lambda d\lambda''\, P(\lambda''; z) \right] \tag{37}$$

where we have used $\lim_{t \to \infty} \lim_{\lambda \to \infty} V(\lambda; t) = \lim_{z \to 0^+} \lim_{\lambda \to \infty} zV(\lambda; z)$. The last formula can be rewritten in the form which reveals the well known[5, 15] *self-averaging* property of the time-asymptotic velocity, when it is nonzero. Indeed, *assuming* that the limit $\lim_{z\to 0^+} \lim_{\lambda\to\infty}\langle P(\lambda; z)\rangle$ is finite, we can write

$$\lim_{t\to\infty} \lim_{\lambda\to\infty} V(\lambda; t) = \lim_{z\to 0^+} \lim_{\lambda\to\infty} z \int_0^\lambda d\lambda' \exp[-z(\lambda-\lambda')\langle P(\lambda; z)\rangle] \times \exp\left\{-z\int_{\lambda'}^{\lambda} d\lambda'' \left[P(\lambda''; z) - \langle P(\lambda; z)\rangle\right]\right\} \tag{38}$$

Due to the above assumption, if $\lambda \to \infty$, the integral in the second exponent represents a random variable which is finite (for high enough λ'', the typical trajectory of $P(\lambda'', z)$ swings around the mean value $\langle P(\lambda; z)\rangle$). Thereupon, for small z, the second exponential tends to unity and the remaining integration yields the reciprocal value of the non-random number (34). Thus, when the mean trapping time τ is finite, the asymptotic velocity is a self-averaging quantity equal to τ^{-1}.[4] If the mean trapping time diverges, the asymptotic velocity vanishes. In this latter case, the disorder-averaged time-asymptotic mean position either tends to a constant (for a negative mean force),or increases slower than linearly.

For the sake of completeness, the *second* thermally-averaged moment $N(\lambda; z) = \int_0^\lambda dx\, x^2 P(x, 0; z)$ can be also connected to the probability density at the origin. Actually, first, the Fokker–Planck equation implies $N(\lambda; z) = (2/z) \int_0^\lambda dx\, xJ(x, 0; z)$. Thereupon, on deriving this expression, we get

$$\frac{d}{d\lambda} N(\lambda; z) = 2M(\lambda; z) - zP(\lambda; z)\, N(\lambda; z), \qquad N(0; z) = 0 \tag{39}$$

Summing up, the first and the second thermally-averaged moments obey a system of stochastic differential equations (36), (39), with $P(\lambda; z)$ playing the role of the "input" noise.

3.3. Dichotomic Random Force

Let the forces in the individual segments assume alternately just two values, $F_\pm = F_0 f_\pm$. We shall always take $f_- < f_+$; the equality would imply a position-independent constant force. Let the lengths of the constant-force segments be independent random variables. The generic probability density for the (dimensionless) lengths of the constant-force

[4] For the exact proof of this statement one needs the simultaneous probability density $\psi(p, w, \lambda; z)$ for the random variables $P(\lambda; z)$ and $W(\lambda; z) = z^2 M(\lambda; z)$. Assuming the condition in the text, one has to prove $\lim_{s\to 0^+} \lim_{\lambda\to\infty} \int \psi(p, w, \lambda; z)\, dp = \delta(w - \tau^{-1})$.

segments will be taken of the form $\rho_{\pm}(\lambda)=n_{\pm}\exp(-\lambda n_{\pm})$, where $1/n_{\pm}$ denotes the mean length of the segments with the force $f_{\pm}$. Due to this assumption, the resulting four-parameter stochastic process $\phi(\lambda)$ is Markovian and it usually referred to as the *asymmetric dichotomic noise.*[11] We shall always work with a *stationary* dichotomic noise. We can set

$$\mu \overset{\text{def}}{=} \langle \phi(\lambda) \rangle = \frac{f_- n_+ + f_+ n_-}{n_- + n_+}, \qquad \langle \phi(\lambda)\, \phi(\lambda') \rangle = \frac{\sigma}{\lambda_c} \exp\left(-\frac{|\lambda-\lambda'|}{\lambda_c}\right) \tag{40}$$

where we have introduced the intensity $\sigma \overset{\text{def}}{=} n_- n_+ (f_+ - f_-)^2/(n_- + n_+)^3$, and the correlation length $\lambda_c \overset{\text{def}}{=} (n_- + n_+)^{-1}$. The statistical properties of the stationary noise are invariant with respect to the inversion $\lambda \to -\lambda$ and to the translation $\lambda \to \lambda - l$.

Let us now focus on Eq. (31). It is convenient to associate with the variable $Q(\lambda; z)$ an overdamped motion of a hypothetical particle. While the "time" λ increases, this particle moves alternately under the influence of the "forces"

$$K_{\pm}(q; z) = -q^2 - f_{\pm} q + z = -[q - q_{\pm}(z)][q - q'_{\pm}(z)] \tag{41}$$

where we have introduced the four quantities

$$q_{\pm}(z) = \sqrt{z + f_{\pm}^2/4} - f_{\pm}/2, \qquad q'_{\pm}(z) = -\sqrt{z + f_{\pm}^2/4} - f_{\pm}/2 \tag{42}$$

Notice the ordering $q'_+(z) < q'_-(z) < 0 < q_+(z) < q_-(z)$, valid for any real positive z and for any values of the parameters $f_- < f_+$. The corresponding "potentials" $U_{\pm}(q; z) = -\int K_{\pm}(q; z)\, dq$ display minima at $q_{\pm}(z)$ and maxima at $q'_{\pm}(z)$. Starting from its initial "position" at infinity, the particle always slides either towards $q_-(z)$ or towards $q_+(z) < q_-(z)$. For any fixed λ the particle can only be found between the coordinate valid for the potential $U_+(q; z)$ and that valid for $U_-(q; z)$. Accordingly, for an arbitrary nonzero z, the probability density $\kappa(q, \lambda; z)$ vanishes outside the finite interval

$$\left[\frac{q_+(z)\, e^{\lambda q_+(z)} - q'_+(z)\, e^{\lambda q'_+(z)}}{e^{\lambda q_+(z)} - e^{\lambda q'_+(z)}}, \frac{q_-(z)\, e^{\lambda q_-(z)} - q'_-(z)\, e^{\lambda q'_-(z)}}{e^{\lambda q_-(z)} - e^{\lambda q'_-(z)}}\right] \tag{43}$$

and it displays two δ function contributions at the edges of this support, their weights being $n_{\mp} \exp(-\lambda n_{\pm})/(n_- + n_+)$. This singular part describes an exponentially decreasing probability of having just one segment in the whole interval of length λ. Obviously, in the limit $\lambda \to \infty$, the support is simply $[q_+(z), q_-(z)]$ and the singular part is missing.

In order to solve Eq. (31) for the dichotomic noise in question we follow the standard steps as described in ref. 11. First, we introduce the joint densities

$$\kappa_{\pm}(q, \lambda; z)\, dq = \mathrm{Prob}\{Q(\lambda; z) \in (q, q+dq) \text{ and } \phi(\lambda) = f_{\pm}\} \tag{44}$$

One has $\kappa(q, \lambda; z) = \kappa_{-}(q, \lambda; z) + \kappa_{+}(q, \lambda; z)$. These densities obey the coupled partial differential equations

$$\frac{\partial}{\partial \lambda}\begin{bmatrix} \kappa_{-}(q, \lambda; z) \\ \kappa_{+}(q, \lambda; z) \end{bmatrix} = -\frac{\partial}{\partial q}\begin{bmatrix} K_{-}(q; z)\, \kappa_{-}(q, \lambda; z) \\ K_{+}(q; z)\, \kappa_{+}(q, \lambda; z) \end{bmatrix} - \begin{bmatrix} n_{-} & -n_{+} \\ -n_{-} & n_{+} \end{bmatrix}\begin{bmatrix} \kappa_{-}(q, \lambda; z) \\ \kappa_{+}(q, \lambda; z) \end{bmatrix} \tag{45}$$

We are looking for the stationary solution $\kappa_{\pm}(q; z) = \lim_{\lambda \to \infty} \kappa_{\pm}(q, \lambda; z)$.[5] Hence we remove the λ-derivative on the l.h.s. of Eq. (45). Introducing the two new functions

$$\xi(q; z) = \frac{K_{-}(q; z)\, K_{+}(q; z)}{n_{-} K_{+}(q; z) + n_{+} K_{-}(q; z)}\, [n_{-} \kappa_{-}(q; z) - n_{+} \kappa_{+}(q; z)] \tag{46}$$

$$\eta(q; z) = K_{-}(q; z)\, \kappa_{-}(q; z) + K_{+}(q; z)\, \kappa_{+}(q; z) \tag{47}$$

and carrying out the corresponding substitution in Eq. (45), we arrive at two independent equations:

$$\frac{d}{dq}\eta(q; z) = 0, \qquad \frac{1}{\xi(q; z)}\frac{d}{dq}\xi(q; z) = -\left(\frac{n-}{K_{-}(q; z)} + \frac{n_{+}}{K_{+}(q; z)}\right) \tag{48}$$

Hence the function $\eta(q; z)$ is simply a constant, which in fact is equal to zero. Actually, Eq. (41) yields $K_{\pm}[q_{\pm}(z); z] = 0$, and conservation of probability entails $\kappa_{\pm}[q_{\mp}(z); z] = 0$. The second differential equation yields

$$\xi(q; z) = \frac{1}{C(z)}\left[\frac{q_{-}(z) - q}{q - q'_{-}(z)}\right]^{\nu_{-}(z)}\left[\frac{q - q_{+}(z)}{q - q'_{+}(z)}\right]^{\nu_{+}(z)},$$

$$\nu_{\pm}(z) = \frac{n_{\pm}}{\sqrt{4z + f_{\pm}^{2}}} \tag{49}$$

[5] The nonstationary case would describe the diffusion on a finite interval, its analysis being very difficult even in a simpler context[25] and of minor physical importance.

where $C(z)$ is a normalization constant. Finally, inverting the transformation $\{(46), (47)\}$, we get $\kappa_{\pm}(q;z) = \mp\xi(q;z)/K_{\pm}(q;z)$ and the stationary density $\kappa(q;z) = \lim_{\lambda\to\infty}\kappa(q,\lambda;z)$ reads

$$\kappa(q;z) = \frac{1}{C(z)}\left\{\frac{1}{[q_-(z)-q][q-q'_-(z)]} + \frac{1}{[q-q_+(z)][q-q'_+(z)]}\right\}$$
$$\times\left[\frac{q_-(z)-q}{q-q'_-(z)}\right]^{\nu_-(z)}\left[\frac{q-q_+(z)}{q-q'_+(z)}\right]^{\nu_+(z)}\Theta[q;q_+(z),q_-(z)] \qquad (50)$$

where we have denoted[6] $\Theta(q;x,y) = \Theta(q-x)\,\Theta(y-q)$. In the final step, the constant $C(z)$ is fixed from the condition $\int_{q_+(z)}^{q_-(z)} \kappa(q;z)\,dq = 1$, that is

$$C(z) = \frac{1}{\langle\sqrt{4z+\phi^2(\lambda)}\rangle}\frac{[q_-(z)-q_+(z)]^{\nu_-(z)+\nu_+(z)}}{[q_+(z)-q'_-(z)]^{\nu_-(z)}[q_-(z)-q'_+(z)]^{\nu_+(z)}}$$
$$\times\mathscr{B}[\nu_-(z),\nu_-(z)]\,\mathscr{F}[\nu_-(z),\nu_+(z),\nu_-(z)+\nu_+(z)+1;-u(z)] \qquad (51)$$

Here $\mathscr{B}(x,y) = \Gamma(x)\,\Gamma(y)/\Gamma(x+y)$ denotes the Euler beta function, $\mathscr{F}(a,b,c;x)$ is the Gauss hypergeometric function,[26] and we have used the abbreviation

$$u(z) = \frac{[q_-(z)-q_+(z)][q'_+(z)-q'_+(z)]}{[q'_+(z)-q'_+(z)][q'_+(z)-q'_-(z)]} \qquad (52)$$

Equations (50) and (51) represent the main result of the present section. The corresponding moments can be obtained by the usual integration: the kth stationary moment $\lim_{\lambda\to\infty}\langle Q^k(\lambda;z)\rangle$ is simply given by the ratio $I_k(z)/I_0(z)$, where we have designated

$$I_k(z) \stackrel{\text{def}}{=} \int_{q_+(z)}^{q_-(z)} dq\, q^k\left\{\frac{1}{[q_-(z)-q][q-q'_-(z)]} + \frac{1}{[q-q_+(z)][q-q'_+(z)]}\right\}$$
$$\times\left[\frac{q_-(z)-q}{q-q'_-(z)}\right]^{\nu_-(z)}\left[\frac{q-q_+(z)}{q-q'_+(z)}\right]^{\nu_+(z)} \qquad (53)$$

In the general case, these integrals can be expressed as a linear combination of the Appell functions $\mathscr{F}_1$[28] (the hypergeometric functions of two variables[26, 27]). In the special case, $I_0(z)$ equals to the integration constant (51).

[6] The product of the two Heaviside functions guarantees the vanishing of the probability density outside the interval $[q_+(z), q_-(z)]$.

4. DISCUSSION

Our general four-parameter description of the dichotomic force provides a rich spectrum of special regimes, which can be analyzed using Eq. (50), where, as indicated above, the probability density $\kappa(q; z)$ stands for $\lim_{\lambda \to \infty} \kappa(q, \lambda; z)$. In the same way, we shall use the simpler designations $Q(z)$ $\pi(p; z)$ and $P(z)$ for the stationary values $Q(\lambda; z)$, $\pi(p, \lambda; z)$ and $P(\lambda; z)$.

4.1. Both Forces Are Negative ($f_- < f_+ < 0$)

The mean force μ in Eq. (40) is negative and the potential consists of segments with a positive slope (c.f. Fig. 2). For any arbitrary realization of the quenched noise, the particle cannot escape to infinity—it can be found with probability one in a *finite* region. Intuitively, one expects a nonzero time-asymptotic mean value of the probability density at the origin, and a finite time-asymptotic value of the thermally-averaged mean position.

Consider the small-z limit of the probability density (50). First, one has $\lim_{z \to 0^+} \nu_\pm(z) = n_\pm / |f_\pm|$. Second, the small-$z$ limits of the expressions

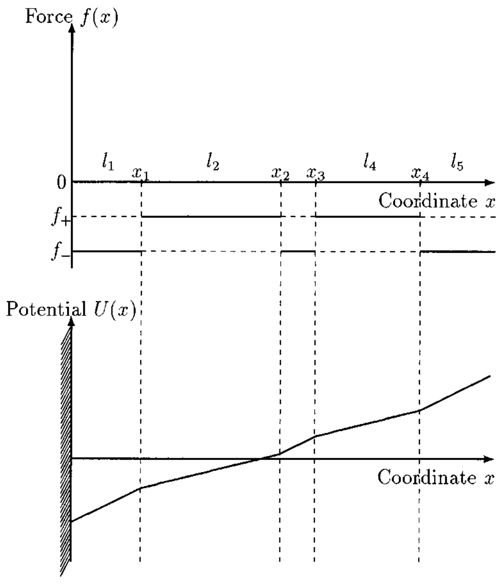

Fig. 2. Typical realization of the Markovian Poisson dichotomic force taking two values of the negative signs, $f_- < f_+ < 0$ with $|f_-| = 2\,|f_+|$. The slope of the corresponding potential is always positive. There are not raps in this case, the particle being stuck towards the reflecting boundary at the origin.

$q_{\pm}(z)$ are $|f_{\pm}|$, whereas $q'_{\pm}(z)$ behave as $-z/|f_{\pm}|$. Analyzing the expressions in (43), the support of the probability density $\lim_{z\to 0^+} \kappa(q; z)$ is the interval $[|f_+|, |f_-|]$. Finally, we have $u(z) \to 0$, i.e., the hypergeometric function in Eq. (51) tends to unity. On collecting these observations, one gets

$$\lim_{z\to 0^+} \kappa(p; z) = |\mu| \frac{|f_-|^{n_+/|f_+|}\, |f_+|^{n_-/|f_-|}}{(|f_-| - q)^{(n_-/|f_-|)-1}\, (q - |f_+|)^{(n_+/|f_+|)-1}} \times \mathscr{B}^{-1}\left(\frac{n_-}{|f_-|}, \frac{n_+}{|f_+|}\right) \times \frac{(|f_-| - q)^{(n_-/|f_-|)-1}\, (q - |f_+|^{(n_+/|f_+|)-1})}{q^{(n_-/|f_-|)+(n_+/|f_+|)+1}} \times \Theta(q; |f_+|, |f_-|) \tag{54}$$

The corresponding moments $\lim_{z\to 0^+}\langle Q^k(z)\rangle$ are all finite and they can be computed analytically by direct integration. In particular for $k=1$ one gets $\lim_{z\to 0^+}\langle Q(z)\rangle = |\mu|$. Thus the time-asymptotic mean value $\lim_{t\to\infty}\langle P(0, 0; t)\rangle$ in the semi-infinite line is seen to be equal to the absolute value of the mean force, $|\mu|$. Note that this result cannot be obtained from the solution of the corresponding free-diffusion model on an infinite line (i.e., without the reflecting boundary condition at the origin). The higher moments are not so simply related to the properties of the random force. Further, in the present case, the mean trapping time (34) is infinite and the time-asymptotic velocity vanishes. The particle is in some sense stuck to the origin. If $f_- < f_+ = 0$, a slightly more complicated calculation reveals the same general conclusions.

4.2. Forces Are of Different Signs ($f_- < 0 < f_+$)

The potential forms a system of traps (c.f. Fig. 1). The traps can only be efficient if the ration n_+/f_+ is comparable with the ratio $n_-/|f_-|$. Otherwise, they are typically "shallow" and they do not represent sufficiently effective obstacles for the particle motion. The "trap-permeability" parameter, as defined by $\theta \stackrel{\text{def}}{=} n_-/|f_-| - n_+/f_+$, will play an important role in the following discussion.[21, 22] In fact, it is proportional to the mean force: $\theta = \mu(n_- + n_+)/|f_-|\, f_+$. Having fixed the forces $f_{\pm}$, the mean force μ can be either positive or negative, depending on the parameters $n_{\pm}$, the value $\mu = 0$ separating two regions with essentially different time-asymptotic dynamics.

Let us consider again the small-z limit of Eq. (50). The quantities $q_-(z)$ and $q'_-(z)$ behave as in the preceding subsection. Presently, however, one has $q_+(z) \sim z/f_+$ and $q'_+(z) \to -f_+$. Thereupon, the small-z limit of the support (43) is now the interval $[0, |f_-|]$. Further, the variable (52) diverges and one has to use the analytic continuation of the hypergeometric function in Eq. (51).

4.2.1. Negative Mean Force ($\mu < 0$). In the small-z limit the normalization constant $C(z)$ alone converges to a finite number and we can safely carry out this limit separately in $C(z)$ and in the rest of the expression (50). The result reads

$$\begin{aligned}\lim_{z\to 0^+} \kappa(q; z) = &\frac{n_+}{n_- + n_+} \frac{f_+^{n_-/|f_-|}(|f_-| + f_+)^{(n_+/f_+)-(n_-/|f_-|)+1}}{|f_-|^{(n_+/f_+)-1}}\\ &\times \frac{1}{\mathscr{B}\left(\dfrac{n_-}{|f_-|}, \dfrac{n_+}{f_+} - \dfrac{n_-}{|f_-|}\right)}\\ &\times q^{(n_+/f_+)-(n_-/|f_-|)-1}(|f_-| - q^{(n_-/|f_-|)-1})\\ &\times (q + f_+)^{-(n_+/f_+)-1}\, \Theta(q; 0, |f_-|)\end{aligned} \tag{55}$$

All the moments of this limiting density exist and can be computed analytically. Let us just quote the result for k: $1 = \lim_{z\to 0^+}\langle Q(z)\rangle = |\mu|$, i.e., we have again $\lim_{t\to\infty}\langle P(0, 0; t)\rangle = |\mu|$. The mean trapping time diverges and the time-asymptotic velocity vanishes. As compared to the previous subsection, the presence of traps does not modify the *modus* of the asymptotic dynamics.

4.2.2. Zero Mean Force ($\mu = 0$). In this Sinai-like case, the small-z analysis of the general expression for the first moment $\langle P(z)\rangle$ together with the Tauber theorem for the inverse Laplace transformation[29] yield the logarithmic decay

$$\langle P(0, 0; t)\rangle \overset{t\to\infty}{\approx} \frac{|f_-|\, f_+}{n_- + n_+} \frac{1}{\log t} \tag{56}$$

4.2.3. Positive Mean Force ($\mu > 0$). In the small-z limit, the normalization constant $C(z)$ diverges as $z^{-\theta}$, $\theta > 0$. The limiting density (50) is concentrated at one point: $\lim_{z\to 0^+} \kappa(q; z) = \delta(q)$. All the moments $Q^k(z)$ are self-averaging quantities, their (non-random) limiting value being zero. On the other hand, the density $\lim_{z\to 0^+}\pi(p;z)$ is a well behaved function, concentrated on the interval $[f_+^{-1}, +\infty[$. Actually, if we first handle the substitution $\pi(p;z)=z\kappa(zp;z)$ in Eq. (50) and *afterwards* exercise the small-z limit, we get

$$\begin{aligned}\lim_{z\to 0^+}\pi(p;z) = {} & \frac{n_-}{n_-+n_+}\,\frac{(|f_-|+f_+)^{(n_-/|f_-|)-(n_+/f_+)+1}}{|f_-|\,f_+^{(n_-/|f_-|)-(n_+/f_+)}}\\ & \times \mathscr{B}^{-1}\left(\frac{n_+}{f_+},\frac{n_-}{|f_-|}-\frac{n_+}{f_+}\right)\\ & \times p\left(p+\frac{1}{|f_-|}\right)^{-(n_-/|f_-|)-1}\left(p-\frac{1}{f_+}\right)^{(n_+/f_+)-1}\\ & \times \Theta\left(p;\frac{1}{f_+},+\infty\right)\end{aligned}\tag{57}$$

Here we come to an essential conclusion[22]: the moment $\lim_{z\to 0^+}\langle P^k(z)\rangle$ is only finite if $k<\theta$. Specifically, for $\theta\in\,]0,1[$, the limit $\lim_{z\to 0^+}\langle P(z)\rangle$ is infinite, i.e., the mean trapping time is also infinite and the time-asymptotic velocity is zero. More precisely, using again the Tauber theorem, we get

$$\langle P(0,0;t)\rangle \overset{t\to\infty}{\approx} \frac{n_-}{n_+(n_-+n_+)}\,\frac{\Gamma^2\left(\dfrac{n_-}{|f_-|}\right)}{\Gamma^2\left(\dfrac{n_+}{f_+}\right)\Gamma(\theta)}\,\frac{f_+^{2(1-\theta)}(|f_-|+f_+)^{2\theta}}{|f_-|^{2\theta}}\,\frac{1}{t^\theta}\tag{58}$$

If $\theta\geqslant 1$, the first moment $\lim_{z\to 0^+}\langle P(z)\rangle$ is finite and its reciprocal gives the (self-averaging) time-asymptotic velocity:

$$\lim_{t\to\infty}\lim_{\lambda\to\infty} V(\lambda;t) = (\theta-1)\,\frac{(n_-+n_+)\,|f_-|\,f_+}{(n_-+n_+)^2-n_-\,|f_-|+n_+f_+}\tag{59}$$

However, if $\theta\in[1,2]$, the small-z limit of the second moment $\lim_{z\to 0^+}\langle P^2(z)\rangle$ is infinite. This can be shown to imply the vanishing of the (static) diffusion constant for the disorder averaged dynamics (this quantity is not discussed here).

4.3. Both Forces Are Positive ($0<f_-<f_+$)

In this case, the slope of the potential is always negative: the particle just slides towards infinity. The small-z limit of the stationary probability density for the random variable $P(z)$ again follows from Eqs. (50) and (51). The normalization constant alone tends to zero and one must operate with the whole expression (50). The result reads

$$\lim_{z\to 0^+} \pi(p; z) = \mu \left(\frac{f_- f_+}{f_+ - f_-}\right)^{(n_-/f_-)+(n_+/f_+)-1} \mathscr{B}^{-1}\left(\frac{n_-}{f_-}, \frac{n_+}{f_+}\right)$$
$$\times p\left(\frac{1}{f_-} - p\right)^{(n_-/f_-)-1}\left(p - \frac{1}{f_+}\right)^{(n_+/f_+)-1}$$
$$\times \Theta\left(p; \frac{1}{f_+}, \frac{1}{f_-}\right) \tag{60}$$

Obviously enough, all the moments of this density exist. For $k = 1$, we get

$$\tau = \lim_{z\to 0^+} \langle P(z)\rangle = \frac{n_- f_- + n_+ f_+ + (n_- + n_+)^2}{(n_- + n_+)(n_- f_+ + n_+ f_- + f_- f_+)} \tag{61}$$

As expected, the mean trapping time is finite. The time-asymptotic velocity is self-averaging and equals the reciprocal of the above expression.[21, 22] There is no anomalous dynamical phase in this case.

4.4. White Shot-Noise Limit

Originally, the dichotomic quenched force has been described by four parameters, $n_\pm \geqslant 0$ and $f_\pm$. Another convenient equivalent four-parameter set is the mean force μ, the intensity σ, the correlation length $\lambda_c = 1/(n_- + n_+)$, already introduced in Eq. (40), and the "non-Gaussianity" parameter[24] $\gamma = |f_+ - f_-|/(n_- + n_+)$, the meaning of which being explained below. It is well known[11, 24] that an appropriate limit of the dichotomic noise yields the Poisson white shot-noise. Actually, consider the parametrization

$$f_- = -\frac{\sigma - \gamma\mu}{\gamma}, \qquad f_+ = \xi, \qquad n_- = \frac{\sigma}{\gamma^2}, \qquad n_+ = \frac{\xi}{\gamma} \tag{62}$$

If we increase the parameter ξ, the force f_+ increases and the mean length of the segments with the force f_+ tends to zero. In the limit $\xi \to \infty$, the parameters μ, σ, and γ keep their values, whereas the correlation length λ_c tends to zero. The limiting form of the correlation function in Eq. (40) is $2\sigma\delta(\lambda - \lambda')$. After the indicated limit, the quenched force displays an array of randomly positioned δ-impulses on the constant background

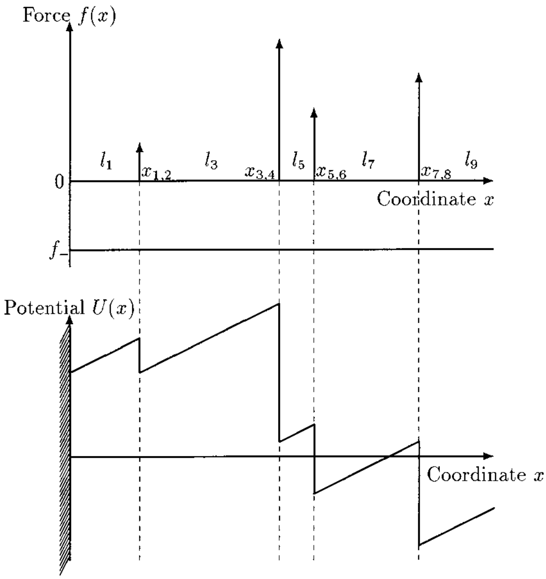

Fig. 3. Typical realization of the white shot-noise random force and of the corresponding potential. The lengths of the vertical segments of the potential are independent, exponentially distributed random variables, their mean value being γ. There are traps in this case.

$-(\sigma-\gamma\mu)/\gamma$ (c.f. Fig. 3). The mean (dimensionless) distance between the δ-impulses is γ^2/σ, their weights being *randomly* distributed with the probability density $\gamma^{-1}\exp(-w/\gamma)\,\Theta(w)$. Thus the parameter γ represents the mean weight of the impulses. On the whole, in the present subsection, the random potential is described by the three parameters $\sigma\geqslant 0$, $\gamma\geqslant 0$, and μ. The potential wells only exist if $f_-<0$, i.e., if $\gamma\mu<\sigma$; Fig. 3 illustrates the typical form of the potential in this case.

Let us now carry out this limiting process in Eqs. (50) and (51). We get $q_+(z)\to 0$ and $\nu_+(z)\to 1/\gamma$, i.e., the support of the density $\kappa(q;z)$ comes to be the interval $[0, q_-(z)]$. The density itself reads

$$\kappa(q;z)=\frac{q_-(z)^{\nu_-(z)1/\gamma}}{[q_-(z)-q'_-(z)]^{\nu_-(z)+1}}\,\mathscr{B}^{-1}[\nu_-(z), 1+1/\gamma]$$
$$\times\mathscr{F}^{-1}\left[\nu_-(z), \nu_-(z)+1, \nu_-(z)+\frac{1}{\gamma}; \frac{q_-(z)}{q_-(z)-q'_-(z)}\right]$$
$$\times q^{1/\gamma}[q_-(z)-q]^{\nu_-(z)-1}\,[q-q'_-(z)]^{-\nu_-(z)-1}\,\Theta[q; 0, q_-(z)] \tag{63}$$

We are again interested in the small-z limit of this probability density and in its moments. We shall restrict the discussion to the physically interesting case with traps, i.e., $\gamma\mu < \sigma$. In this case, the variable of the Gauss hypergeometric function in Eq. (51) tends to 1^- and we use an appropriate analytic-continuation formula.[26]

If $\mu < 0$, the time-asymptotic value of the averaged density at origin is again $\lim_{t \to \infty} \langle P(0, 0; t) \rangle = |\mu|$. If $\mu = 0$ is zero, we observe the logarithmic decay

$$\langle P(0, 0; t) \rangle \overset{t \to \infty}{\approx} \sigma \frac{1}{\log t} \tag{64}$$

If $\mu > 0$, the time-asymptotics is controlled by the trap-permeability parameter θ. Presently, it can be written as $\theta = \mu/(\sigma - \mu\gamma)$. For $\theta \in]0, 1]$, the trapping time diverges and the asymptotic velocity vanishes. More precisely, we have

$$\langle P(0, 0; t) \rangle \overset{t \to \infty}{\approx} \frac{\Gamma^2\left(\theta + \frac{1}{\gamma}\right)}{\Gamma^2\left(\frac{1}{\gamma}\right) \Gamma(\theta)} \frac{\sigma\gamma^{2\theta}}{(\sigma - \mu\gamma)^{2\theta}} \frac{1}{t^\theta} \tag{65}$$

Finally, if $\theta \geqslant 1$, the first moment $\lim_{z \to \in^+} \langle P(z) \rangle$ is finite and its reciprocal gives the (self-averaging) time-asymptotic velocity:

$$\lim_{t \to \infty} \lim_{\lambda \to \infty} V(\lambda; t) = (\theta - 1) \frac{\sigma - \mu\gamma}{1 + \gamma} \tag{66}$$

4.5. Gaussian-White-Noise Limit

As already mentioned in the Introduction, the diffusion process in a Brownian environment represent an archetypal formulation which has been deeply pursued in the literature. On the mathematical side,[30–35] the emphasis has been given, e.g., to the precise formulation of the pertinent limit theorems,[6] to the phenomenon of localization by random centering,[34] and to the extremal properties of the particle's trajectories. In the present subsection, some of these results will be recovered as a particular case of our formulation.

In fact, it is well known[11, 24] that an appropriate limit of the dichotomic noise yields the Gaussian white noise. However, the Gaussian white noise can be also obtained as a limit of the white shot-noise which has been introduced in the preceding subsection: one simply sets $\gamma \to 0^+$.

This means that the mean weight of the δ-impulses of the force tends to zero, and simultaneously their density $n_- = \sigma/\gamma^2$ increases, such that the product (density) $\times$ (mean weight)^2 remains constant. The bias μ and the intensity σ keep their values and the correlation function in Eq. (40) is again $2\sigma\delta(\lambda - \lambda')$. The quenched force displays an infinitely dense array of δ peaks in both directions, their weights being infinitely small. Notice that this limit can only be achieved if we start with the dichotomic random force taking two values of different signs.

The Gaussian-white-noise results simply emerge after we carry out the small-γ limit in Eqs. (63). Particularly, we have $q_-(z) \to \infty$, i.e., the support of the probability density $\kappa(q; z)$ becomes the infinite interval $[0, +\infty[$. The density itself reads

$$\kappa(q; z) = \frac{1}{2} \frac{z^{\theta/2}}{\mathscr{K}_\theta\left(\frac{2}{\sigma}\sqrt{z}\right)} \frac{1}{q^{\theta+1}} \exp\left[-\frac{1}{\sigma}\left(q + \frac{z}{q}\right)\right] \Theta(q; 0, +\infty) \qquad (67)$$

in accordance with the result found in refs. 5 and 16. Presently, the trapper-meability parameter simply measures the ratio between the mean force and its intensity: $\theta = \mu/\sigma$.

The formula (67) is valid for arbitrary values of the parameters σ and μ. If $\mu < 0$, we have again $\lim_{t\to\infty}\langle P(0, 0; t)\rangle = |\mu|$. In the Sinai case, i.e., for $\mu = 0$, the asymptotic behaviour is again given by Eq. (64). The corresponding result for the infinite line without the reflecting boundary condition at the origin is [16] $\langle P(0, 0; t)\rangle \overset{t\to\infty}{\approx} \sigma/(\log t)^2$. Thus the presence of the boundary slows down the decay of the disorder-averaged probability density at the origin. Having $\mu > 0$ and $\theta \in\,]0, 1[$, the asymptotic velocity vanishes and the averaged probability density at the origin decreases algebraically as

$$\langle P(0, 0; t)\rangle \overset{t\to\infty}{\approx} \frac{\sigma^{1-2\theta}}{\Gamma(\theta)} \frac{1}{t^\theta} \qquad (68)$$

For example, $\theta = 1/2$ yields the exact solution $\langle P(0, 0; t)\rangle = 1/\sqrt{\pi t}$, valid for any time. Finally, when $\theta \geqslant 1$, the time-asymptotic disorder-averaged mean position linearly increases, the self-averaging velocity being $(\theta - 1)\,\sigma$. The damping of the disorder-averaged probability density at the origin can be exemplified by taking $\theta = 3/2$: we then get $\langle P(z)\rangle = 2/(\sigma + 2\sqrt{z})$, that is

$$\langle P(0, 0; t)\rangle = \frac{1}{\sqrt{\pi t}} - \frac{\sigma}{2} \exp\left(\frac{1}{4}\sigma^2 t\right) \operatorname{erfc}\left(\frac{1}{2}\sigma\sqrt{t}\right) \overset{t\to\infty}{\approx} \frac{2}{\sigma^2\sqrt{\pi}} \frac{1}{t^{3/2}} \qquad (69)$$

This asymptotic behaviour should be contrasted with the exponential damping which takes place in the presence of a positive homogeneous deterministic force, as found in Subsection 2.1.

5. CONCLUSION

In the present paper, a transfer-matrix-like method for solving diffusion problems in a piecewise linear random potential has been introduced. The formulae for the Green function derived in the second section can be easily adapted to numerical simulation of the diffusive motion in *any* potential of the mentioned type. For example, the force can be assumed to be a semi-Markovian or a non-Markovian variant of the dichotomic noise,[36, 37] it can exhibit jumps of random magnitudes (kangaroo process[29]), etc. For any such process, our analysis is valid up to Subsection 3.3. Our subsequent choice of a Markovian dichotomic process has been dictated by a relatively direct possibility to get the asymptotic solution of the stochastic equations (30), (31).

Let us summarize the preceding discussion. The dynamical effects of the quenched disorder have been evinced by examining the varying stochastic features of a single random variable, namely, the probability density of the particle's occurrence at the origin. Having a negative mean bias, the time-asymptotic and disorder-averaged value of this quantity is proportional to the absolute value of the mean force. In the Sinai-like case, i.e., for the vanishing mean bias, we have given the exact asymptotic formula describing the decay of this quantity. The decay is slower than in the corresponding model without the reflecting boundary at the origin. Finally, having a positive mean bias, the particle escapes towards infinity, with a finite velocity or not, depending on the value of the trap-permeability parameter. The existence of deep traps with a long trapping time is crucial for the existence of anomalous dynamical phases.

In the present work, we have chosen to formulate the diffusion problem in the presence of reflecting boundary conditions. After a slight modification, the method can be adapted to other types of boundary conditions. For instance, taking a fixed probability density at two boundaries, our method can be used to the analysis of the stationary-flux distribution in the one-dimensional random medium.[21]

We have not aimed at the exhaustive description of the particle dynamics. Instead, we have concentrated on features which can be directly related to the probability density at the origin. The detailed description of the "noise" $P(\lambda; z)$ allows, at least in principle, for an investigation of other aspects, such as the time-asymptotic thermally-averaged first moment of the particle's position. Another example would be the disorder-averaged diffusion coefficient. Its analysis requires the calculation of the second thermally-averaged moment of the particle's position on the one hand, and the higher-order (generally non-self-averaging) terms in the small-z expansion of the first moment on the other hand. We have shown that the thermally-averaged moments obey a system of stochastic differential equations with $P(\lambda; z)$ playing the role of the "input" noise. The probabilistic description of the moments will be reported elsewhere.

Finally, let us mention a possible correspondence between the space-time continuous model and its discrete analogue. The Fokker–Planck equation with the Gaussian-white-noise quenched disorder (as discussed in Subsection 4.5) coheres with the space-discrete Master Equation including mutually independent and identically distributed random transition rates.[1] The parallel discussion of these two models has been given in ref. 16. However, the space-discrete counterpart of the continuous model with the Poisson random force (as particularized in Subsection 3.3) is considerably more involved: one assumes the Master equation with the spatially correlated random rates, e.g., the random rates themselves form a stationary Markov chain. This model does not seem to be adequately treated in the literature, yet (c.f., however, the discussion of the spatial correlations within the simpler frame of the *directed* random walk in refs. 19 and 20).

Summing up, the paper presents an approximation-free study of the diffusive dynamics in an one-dimensional Markovian Poisson random potential. It provides a firm basis for the intuitive understanding of diffusion in more involved circumstances.

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